

# The Network Utility Maximization Problem with Multiclass Traffic

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## Abstract

The concave utility in the Network Utility Maximization (NUM) problem is only suitable for elastic flows. In networks with multiclass traffic, the utility can be concave, linear, step or sigmoidal. Hence, the basic NUM becomes a nonconvex optimization problem. The current approach utilizes the standard dual-based decomposition method. It does not converge in case of scarce resource. In this paper, we propose an algorithm that always converges to a local optimal solution to the nonconvex NUM after solving a series of convex approximation problems. Our techniques can be applied to any log-concave utilities.

## 1. Introduction

The network utility maximization (NUM) problem with concave utilities for elastic flows, the basic NUM, has been widely studied, e.g., [1–2]. In a network where  $\mathcal{N}$  and  $\mathcal{L}$  are the set of sources and set of links,  $N$  and  $L$  are their respective cardinalities, NUM is stated as follows

**Problem 1:**

$$\begin{aligned} \text{Max.} \quad & \sum_{s \in \mathcal{N}} U_s(x_s) \\ \text{s.t.} \quad & \sum_{s: l \in L(s)} x_s \leq c_l, \forall l \in \mathcal{L} \end{aligned}$$

where rate  $\mathbf{x}$  is bounded in  $[\mathbf{x}^{\min}, \mathbf{x}^{\max}]$ ,  $c_l$  is the

capacity of link  $l$ , and  $L(s)$  is the path of flow with source  $s$ . Using dual-based decomposition method, [2] proposes the dual-based distributed algorithm to find the optimal rate allocation for the network with multiclass services, the utilities can be concave, sigmoidal, linear, or step functions, [3]. Therefore, Problem 1 becomes a nonconvex optimization problem and is difficult to solve. Applying the standard dual-based distributed algorithm to nonconvex NUM can lead to a suboptimal or even infeasible rate allocation [4–5]. The standard dual-based algorithm only converges in case the link's capacity is larger than a critical value. On the other hand, by an indirectly approach, the authors in [6] replace the utility with a new function  $\int 1/U(x) dx$ , which is concave as  $U(x)$

is an increasing function. Then they derive the utility-proportional fairness algorithm. Their solution is a lower bound of the solution to Problem 1.

In this paper, we propose a novel algorithm that can archive the local optimal solution for any link's capacity based on the successive approximation method. The successive convex approximation method is introduced in [7] and has been used mainly in geometric programming for power control problems, i.e., [8]. We utilize this method to approximate the objective to a new concave function using arithmetic-geometric mean inequality and derive the original problem to the convex approximation problem which is solved easily by dual decomposition method. After a series of approximations, the results will converge to the local optimal solution to the original problem.

## 2. Successive approximations method

We replace Problem 1 with the new problem, Problem 2, which has the same optimal rate allocation. The replacement problem is given by

**Problem 2:**

$$\begin{aligned} \text{Max.} \quad & \text{og} \left( \sum_{s \in \mathcal{N}} U_s(x_s) \right) \\ \text{s.t.} \quad & \sum_{s: l \in L(s)} x_s \leq c_l, \forall l \in \mathcal{L}. \end{aligned}$$

We have Problem 1 and Problem 2 share the same local/global optimal rate allocation since  $\log(\cdot)$  is a monotonically increasing function.

Problem 2 remains a nonconvex optimization problem because of the nonconcave objective. However, for the use of the successive approximations method which requires a convex objective for the minimization problem (or a concave objective for the maximization problem) [7], we transform Problem 2 into an epigraph-form equivalent problem [9, p.4.2.4] as

follows.

**Problem 3:**

$$\begin{aligned} \text{Max. } & t \\ \text{s.t. } & 0 \leq \left( \sum_{s \in \mathcal{N}} U_s(x_s) \right) \\ & \sum_{s: l \in L(s)} x_s \leq c_l, \forall l \in \mathcal{L} \end{aligned}$$

**Result 1:** For any vector  $\theta = [\theta_1, \theta_2, \dots, \theta_N] \succ 0$  and

$$\mathbf{1}^T \theta = 1$$

$$\log\left(\sum_{s \in \mathcal{N}} U_s(x_s)\right) \geq \sum_{s \in \mathcal{N}} \theta_s \log\left(\frac{U_s(x_s)}{\theta_s}\right). \quad (1)$$

*Proof:* Making use of arithmetic-geometric mean inequality, we have

$$\sum_{s \in \mathcal{N}} U_s(x_s) \geq \prod_{s \in \mathcal{N}} \left(\frac{U_s(x_s)}{\theta_s}\right)^{\theta_s}.$$

Taking the logarithm of both sides of the inequality, we obtain (1). The equality holds if

$$\theta_s = \frac{U_s(x_s)}{\sum_{k \in \mathcal{N}} U_k(x_k)}, \forall s = 1, \dots, N. \quad (2)$$

From Result 1, we consider the approximate problem with the new approximate constraint

**Problem 4**

$$\begin{aligned} \text{Max. } & t \\ \text{s.t. } & 0 \leq \sum_{s \in \mathcal{N}} \theta_s \left(\frac{U_s(x_s)}{\theta_s}\right) \\ & \sum_{s: l \in L(s)} x_s \leq c_l, \forall l \in \mathcal{L} \end{aligned}$$

Transforming Problem 4 back to the canonical form, we obtain the equivalent problem

**Problem 5:**

$$\begin{aligned} \text{Max. } & \sum_{s \in \mathcal{N}} \tilde{U}_s^{(\theta_s)}(x_s) \\ \text{s.t. } & \sum_{s: l \in L(s)} x_s \leq c_l, \forall l \in \mathcal{L} \end{aligned}$$

where  $a^c \log(x+1), a^c > 0$ .

**Result 2:** With the following considered utilities

- Concave functions:  $a^c \log(x+1), a^c > 0$ ;
- Linear functions:  $a^l x, a^l > 0$ ;

- Sigmoidal functions:  $c^s / (1 + e^{-a^s(x-b^s)})$

functions  $\tilde{U}_s^{(\theta_s)}(x_s), \forall s \in \mathcal{N}$  are concave.

*Proof:* We can easily verify this by taking the second derivative test for all three functions. ■

From Result 2, Problem 5 becomes the basic NUM with a concave objective; therefore, it is solved easily using the dual decomposition method, as in Section II-B. Notice that Problem 5 is an approximation of Problem 2, not Problem 1.

### 3. Solution to the approximation problem

We now apply the standard dual-based decomposition method to Problem 5. The dual function is given by

$$\begin{aligned} D(\lambda) &= \max_{\mathbf{x}} \left( \sum_{s \in \mathcal{N}} \tilde{U}_s^{(\theta_s)}(x_s) - \sum_{l \in \mathcal{L}} \lambda_l \left( \sum_s x_s - c_l \right) \right) \\ &= \sum_{s \in \mathcal{N}} \max_{x_s} \left( \tilde{U}_s^{(\theta_s)}(x_s) - \left( \sum_{l \in L(s)} \lambda_l \right) x_s \right) + \sum_{l \in \mathcal{L}} \lambda_l c_l, \end{aligned} \quad (3)$$

and the dual problem is  $\min_{\lambda \geq 0} D(\lambda)$ .

Let  $q_s = \sum_{l \in L(s)} \lambda_l$ , the rate update is given by

$$x_s(t+1) = [\tilde{U}_s^{(\theta_s)'}{}^{-1}(q_s)]_{x_s^{\min}}^{x_s^{\max}}, \forall s \in \mathcal{N}. \quad (4)$$

where  $[a]_c^b = \max(\min(a, b), c)$ .

Using the gradient projection algorithm for the dual problem, we obtain the congestion price update

$$\lambda_l(t+1) = \left[ \lambda_l(t) + \kappa \left( \sum_{s: l \in L(s)} x_s(t) - c_l \right) \right]^+, \forall l \in \mathcal{L} \quad (5)$$

where stepsize  $\kappa$  is small enough for the convergence of the algorithm and  $[a]^+ = \max(a, 0)$ .

**Algorithm 1:** Successive approximation algorithm

Initialize from a random initial point

- 1) Each source updates  $\theta_s(\tau)$  using (2).
- 2) Each source updates the rate using (4), each link updates its price according to (5) until convergence to a value  $\mathbf{x}^0(\tau)$ .

- 3) Each source calculates its utility  $U_s(x_s^0(\tau))$  and transmits this information to all other

**Theorem 1:** Algorithm 1 increases the aggregate utility after each outer iteration. The stationary point of the algorithm satisfies the Karush-Kuhn-Tucker conditions of Problem 2.

*Proof:* Let  $f(\mathbf{x}) = t / \log\left(\sum_{s \in \mathcal{N}} U_s(x_s)\right)$  and

$$\tilde{f}(\mathbf{x}) = t / \sum_{s \in \mathcal{V}} \theta_s \log(U_s(x_s) / \theta_s). \text{ According}$$

To [7], we need to prove the following three conditions for convergence to the KKT point of the algorithm:

C1)  $f(\mathbf{x}) \leq \tilde{f}(\mathbf{x})$ , C2)  $f(\mathbf{x}^0) = \tilde{f}(\mathbf{x}^0)$ , and C3)  $\nabla f(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^0} = \nabla \tilde{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^0}$ , where  $\mathbf{x}^0$  is the optimal solution of the previous iteration.

Conditions 1) and 2) are clearly satisfied with Result 1. Condition 3) is verified by taking the derivative and applying (2). Therefore, Algorithm 1 converges to the local optimal rate allocation of the original problem. ■

#### 4. Numerical Results and Conclusions

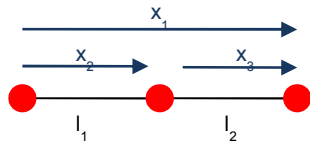


Figure 1: Network topology.

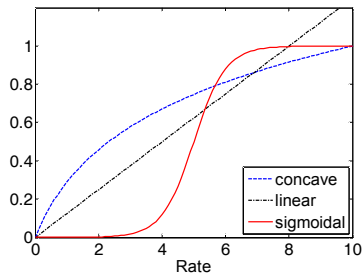


Figure 2: Utility functions.

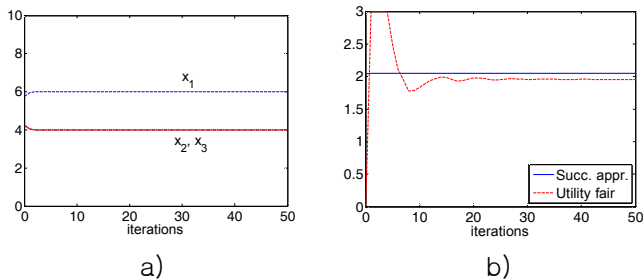


Figure 3: User rate and aggregate utility evolutions. We use the same network topology as that in [6]. The network has two links and three flows as described in Figure 1. The utility functions of the flows are  $1/(1 + \exp(-2(x_2 - 5)))$  (sigmoidal),  $0.125x_2$  (linear), and  $\log(x_3 + 1)/\log(11)$  (concave), respectively (see Figure 2. These functions are normalized at  $\mathbf{x}^{\max}=[10;8;10]$  Mbps.  $\mathbf{x}^{\min}=0$  Mbps and  $\mathbf{c}=10$  Mbps.

After running Algorithm 1, the convergence of rate and utility are shown in Figures 3a and b, where the

optimal rate allocation is  $\mathbf{x}^*=[6.0047;3.9953;3.9953]$  Mbps. We can see that the proposed algorithm has higher aggregate utility than does the utility-fair one ( $U^*=2.052$  and  $U^{U^{fair}}=1.9559$ ).

We've addressed the nonconvex NUM with multiclass traffic with sigmoidal, linear, and concave utilities. Our algorithm based on the successive convex approximation method that converges to a locally optimal rate allocation for any link capacity. The simulation shows the convergence of the algorithm and the higher aggregate utility compared to that of the utility-proportional-fair rate allocation.

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