The Successive Approximation Approach for Multi-path Utility Maximization Problem

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***Abstract*—In this paper, we solve the network utility maximization (NUM) problem for networks with both multi-path and single-path users. To deal with the non-strictly convexity and non-separability of the problem, we approximate it to a new strictly convex and separable problem which is efficiently solved by the standard dual-based decomposition approach. After a sequence of approximations, the solution to the approximation problem converges to a globally optimal solution of the original NUM. From the theoretical analysis, we also introduce a design of multi-path Reno (mReno) based on the reverse engineering framework of TCP Reno. The fairness among multi-path users and single-path users is guaranteed.**

I. Introduction

In the network environment where the multi-path users and single-path users are coexisting, the network utility maximization (NUM) problem is a non-strictly convex and non-separable optimization problem. In many previous works on multi-path NUM [1]–[6], the authors perform either subtraction or addition of a strictly convex or concave function to the objective in order to transform Main Problem to a strictly convex problem. This new problem is solved distributively by primal approach or dual approach. The result is a globally optimal solution. However, the new strictly convex problem remain non-separable, so these previous mentioned works do not fully model the case of a multi-path user having paths with different characteristics, for example, different round-trip-times. On the other hand, the current TCPs are window-based update protocols whereas the algorithms in [1], [3]–[6] are rate-based updates, hence, it is quite difficult to deploy them to the current Internet.

In this paper, we approximate the original problem to a new strictly convex and separable problem. After a series of approximations, the solution to the approximation problem which is obtained by dual-based approach converges to a globally optimal solution of the original problem. Our algorithm is distributively implemented. To adapt the rate allocation on each path, each source depends on the local information, which is the total rate of all paths and the congestion-price feedbacks of paths. Moreover, by utilizing the separability of the new approximation problem and the proposed dual-based algorithm, we develop a multi-path Reno protocol based on the reverse engineering of TCP Reno. The multi-path Reno presented in this paper is totally compatible to TCP Reno. The multi-path users running multi-path Reno can fairly cooperate with the single-path users running TCP Reno in a network.

The successive approximation approach is introduced in [7] and it is usually applied to geometric programming in power control problems, such as [8], [9], to approximate the non-convex capacity constraints. [8] has an interesting overview about this method. Also using the same method, the authors in [10] approximate both objective and power constraints of the NUM to jointly control the power and rate in a multihop wireless network with multiclass traffic. However, before the approximation, the problem is transformed into an epigraph form in order to have a new problem with concave objective and non-convex constraints as the problem defined in [7]. In this paper, we approximate the NUM with non-strictly concave objective directly into a new strictly convex problem. So, the proof of convergence is given in the paper. The rest of the paper is organized as follows: Section II presents the approximation problem, the successive approximation algorithm, and its convergence in fluid model. Section III introduces the multi-path TCPs design framework and an example of multi-path Reno . And finally, the Multi-path Reno experiments and conclusions are presented in Sections IV and V, respectively.

Notations: We use italic characters to denote variables and bold characters to denote vectors. For example, $\mathbf{x}_s \triangleq [x_{s,1}, x_{s,2}, ...]^T$ is the rate vector of all subflows from source s, and $\mathbf{x} \triangleq [\mathbf{x}_1^T, \mathbf{x}_2^T, ..., \mathbf{x}_N^T]^T$ is rate vector of all subflows from all sources. Similarly, $\theta_{s,i}$ is the auxiliary variable associated with subflow i of source s, $\boldsymbol{\theta}_s \triangleq [\theta_{s,1}, \theta_{s,2}, ...]^T$, and $\boldsymbol{\theta} \triangleq [\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T, ..., \boldsymbol{\theta}_N^T]^T$.

II. Analysis

A. Approximation problem

Consider a communication networks with the set of sources $\mathcal{N}$ and set of links $\mathcal{L}$. The network utility maximization

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(NUM) framework for multi-path is stated as follows

$$\text{Main Problem}:$$
$$\begin{aligned}\text{Max. } & \sum_{s\in\mathcal{N}} U_s\Big(\sum_{i=1}^{R_s} x_{s,i}\Big)\\ \text{s.t. } & \sum_{s\in\mathcal{N}}\sum_{i:l\in\mathcal{R}_{s,i}} x_{s,i}\le c_l,\ \forall l\in\mathcal{L}.\end{aligned}$$

$U_s(\sum_{i=1}^{R_s} x_{s,i})$ is a non-strictly concave function associated with source/flow s. $\mathcal{R}_{s,i}$ is the i-th path from source s, $\mathcal{R}_s \triangleq \{\mathcal{R}_{s,1},\mathcal{R}_{s,2},...\}$ is the set of paths associated with source s, R_s is number of subflows from source s, and $x_{s,i}$ is the allocated rate on i-th subflow of source s. We assume $x_{s,i}\in[x_{s,i}^{\min},x_{s,i}^{\max}]$.

We now derive the inequality to approximate Main Problem. It is known that if $f(.)$ is a concave function, the Jensen's inequality $f(\sum_{i=1}^{R_s}\theta_{s,i}z_i) \ge \sum_{i=1}^{R_s}\theta_{s,i}f(z_i)$ holds for all $\boldsymbol{\theta}_s \succ \mathbf{0}$ and $\mathbf{1}^T\boldsymbol{\theta}_s = 1$. After replacing $x_{s,i}=\theta_{s,i}z_i$, we obtain the following inequality

$$U_s\Big(\sum_{i=1}^{R_s} x_{s,i}\Big)\ge\sum_{i=1}^{R_s}\theta_{s,i}U_s\Big(\frac{x_{s,i}}{\theta_{s,i}}\Big). \quad (1)$$

Note that the equality of (1) holds if

$$\theta_{s,i}=\frac{x_{s,i}}{\sum_{j=1}^{R_s}x_{s,j}},\forall i=1,..,R_s,s=1,..,N. \quad (2)$$

By denoting $\tilde{U}_{s,i}(x_{s,i};\theta_{s,i})\triangleq\theta_{s,i}U_s\big(\frac{x_{s,i}}{\theta_{s,i}}\big)$, a function of $x_{s,i}$ parameterized by $\theta_{s,i}$, we have the approximation of Main Problem as follows

$$\text{Approximation Problem}:$$
$$\begin{aligned}\text{Max. } & \sum_{s=1}^{N}\sum_{i=1}^{R_s}\tilde{U}_{s,i}(x_{s,i};\theta_{s,i})\\ \text{s.t. } & \sum_{s=1}^{N}\sum_{i:l\in\mathcal{R}_{s,i}} x_{s,i}\le c_l,\ \forall l\in\mathcal{L}.\end{aligned}$$

Approximation Problem is exactly the basic NUM problem in which a new separate and strictly concave utility is associated with each subflow. Therefore, the network treats each subflow as a separate flow. Now, we can solve Approximation Problem by the standard dual decomposition method as in [11].

The dual function is given by

$$\begin{aligned}D(\boldsymbol{\lambda}) =& \max_{\mathbf{x}}\Bigg(\sum_{s=1}^{N}\sum_{i=1}^{R_s}\tilde{U}_{s,i}(x_{s,i};\theta_{s,i})\\ &-\sum_{l=1}^{L}\lambda_l\Big(\sum_{s=1}^{N}\sum_{i:l\in\mathcal{R}_{s,i}}x_{s,i}-c_l\Big)\Bigg) \quad (3)\\ =&\sum_{s=1}^{N}\sum_{i=1}^{R_s}\max_{x_{s,i}}\Bigg(\tilde{U}_{s,i}(x_{s,i};\theta_{s,i})-\Big(\sum_{l\in\mathcal{R}_{s,i}}\lambda_l\Big)x_{s,i}\Bigg)\\ &+\sum_{l=1}^{L}c_l\lambda_l, \quad (4)\end{aligned}$$

and the dual problem is $\min_{\boldsymbol{\lambda}\succeq 0}D(\boldsymbol{\lambda})$. ($N$ and L are the cardinalities of $\mathcal{N}$ and $\mathcal{L}$, respectively.)

Let $q_{s,i}(t)\triangleq\sum_{l\in R_{s,i}}\lambda_l(t)$ be the congestion price of i-th path from source s. Because the subproblem $\max_{x_{s,i}}\Big(\tilde{U}_{s,i}(x_{s,i};\theta_{s,i})-(\sum_{l\in\mathcal{R}_{s,i}}\lambda_l)x_{s,i}\Big)$ is a convex problem, its solution also satisfies the KKT conditions. From the first derivative condition, we have the rate update on each subflow given by

$$x_{s,i}(t)=\Big[\tilde{U}_{s,i}'^{-1}(q_{s,i}(t);\theta_{s,i})\Big]_{x_{s,i}^{\min}}^{x_{s,i}^{\max}}, \quad (5)$$

where $[a]_c^b=\max(\min(a,b),c)$.

Applying the projected gradient algorithm to the dual problem, we obtain the congestion price update of each link as follows:

$$\lambda_l(t+1)=\Big[\lambda_l(t)+\kappa\Big(\sum_{s=1}^{N}\sum_{i:l\in\mathcal{R}_{s,i}}x_{s,i}(t)-c_l\Big)\Big]^+,\ \forall l\in\mathcal{L}, \quad (6)$$

where stepsize κ is sufficiently small for the convergence of the algorithm and $[a]^+=\max(a,0)$.

B. Successive approximation algorithm for multi-path NUM

Algorithm 1: Initialize from any feasible point, for example, $\mathbf{x}=0$ and $\theta_{s,i}=\frac{1}{R_s}$. In the τ-th iteration,

1) Each source updates $\theta_{s,i}$ according to (2) which $\mathbf{x}_s$ is the result of the previous iteration;
2) With updated $\boldsymbol{\theta}_s$, source s updates the transmit rate of its subflows according to (5), and links update their prices according to (6) until convergence;
3) Increase τ and go back to step 1.

Theorem 1: Algorithm 1 converges and the stationary point satisfies the Karush-Kuhn-Tucker conditions of Main Problem.

Proof: We define some parameters for convenience as follows:

- $\mathbf{x}^o(\tau)$, the initial point of step τ;
- $\mathbf{x}^\infty(\tau)$, the stationary point of step τ;
- $G(\mathbf{x})\triangleq\sum_{s=1}^{N}U_s\big(\sum_{i=1}^{R_s}x_{s,i}\big)$; and
- $\tilde{G}(\mathbf{x};\boldsymbol{\theta})\triangleq\sum_{s=1}^{N}\sum_{i=1}^{R_s}\tilde{U}_{s,i}\big(x_{s,i};\theta_{s,i}\big)$, the function of $\mathbf{x}$ parameterized by $\boldsymbol{\theta}$.

First, we prove the convergence of the algorithm. The solution to Approximation Problem indeed monotonically increases the objective of Main Problem in each step:

$$\begin{aligned}G(\mathbf{x}^\infty(\tau-1)) &= \tilde{G}(\mathbf{x}^o(\tau);\boldsymbol{\theta}(\tau)) & (7)\\ &\le\tilde{G}(\mathbf{x}^\infty(\tau);\boldsymbol{\theta}(\tau)) & (8)\\ &\le G(\mathbf{x}^\infty(\tau)). & (9)\end{aligned}$$

(7) is obtained by replacing $\theta_{s,i}(\tau)=\frac{x_{s,i}^\infty(\tau-1)}{\sum_{j=1}^{R_s}x_{s,i}^\infty(\tau-1)}$ and $\mathbf{x}^o(\tau)=\mathbf{x}^\infty(\tau-1)$, (8) is satisfied because $\mathbf{x}^\infty(\tau)$ is an optimal point given $\boldsymbol{\theta}(\tau)$, and (9) is directly from (1). Moreover, $G(\mathbf{x})$ is always bounded since $\mathbf{x}$ is bounded, therefore, Algorithm 1 converges.

We now prove that the stationary point of Algorithm 1 is also the KKT point of Main Problem. Define $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ be the solution to Approximation Problem along with $\boldsymbol{\theta}^*$. Thus, $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is also the KKT point of Approximation Problem.

$$\nabla \tilde{G}(\mathbf{x}^*; \boldsymbol{\theta}^*) - \mathbf{q}^* = 0, \tag{10}$$

$$\lambda_l^* (\sum_{s=1}^{N} \sum_{i: l \in \mathcal{R}_{s,i}} x_{s,i}^* - c_l) = 0, \forall l \in \mathcal{L}, \tag{11}$$

$$\sum_{s=1}^{N} \sum_{i: l \in \mathcal{R}_{s,i}} x_{s,i}^* \leq c_l, \tag{12}$$

$$\lambda_l^* \geq 0, \tag{13}$$

where $\mathbf{q} = [\mathbf{q}_1^T, \mathbf{q}_2^T, ..., \mathbf{q}_N^T]^T$ is congestion price column vector of all paths of every sources.

We prove that $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ also satisfies the KKT conditions of Main Problem. Observe that

$$\begin{aligned}
\frac{\partial G(\mathbf{x})}{\partial x_{s,1}}\Big|_{\mathbf{x}_s = \mathbf{x}_s^*} &= \frac{\partial U_s(\sum_{i=1}^{R_s} x_{s,i})}{\partial x_{s,i}}\Big|_{\mathbf{x}_s = \mathbf{x}_s^*} \\
&= \frac{\partial U_s(x_{s,1} + \sum_{i=2}^{R_s} x_{s,i}^*)}{\partial (x_{s,1} + \sum_{i=2}^{R_s} x_{s,i}^*)}\Big|_{x_{s,1} = x_{s,1}^*} \\
&= \frac{\partial U_s(x)}{\partial x}\Big|_{x_s = \sum_{i=1}^{R_s} x_{s,i}^* = \frac{x_{s,1}^*}{\theta_{s,1}^*}} \\
&= \frac{\partial U_s(\frac{x_{s,1}}{\theta_{s,1}^*})}{\partial (\frac{x_{s,1}}{\theta_{s,1}^*})}\Big|_{x_{s,1} = x_{s,1}^*} \\
&= \frac{\partial \tilde{G}(\mathbf{x}; \boldsymbol{\theta})}{\partial x_{s,1}}\Big|_{\mathbf{x}_s = \mathbf{x}_s^*},
\end{aligned}$$

and similarly, it is the same for the proof of all other partial differential equations. Therefore, $\nabla \tilde{G}(\mathbf{x}^*; \boldsymbol{\theta}^*) - \mathbf{q}^* = \nabla G(\mathbf{x}^*) - \mathbf{q}^* = 0$. The remaining conditions are kept the same. Thus the second statement is proved. ■

Main Problem is a convex optimization problem even though the objective is not strictly concave. So the KKT point is also a global optimum of the optimization problem [13]. As a result, Algorithm 1 converges to a globally optimal rate allocation. Moreover, $\boldsymbol{\theta}_s$ can be updated distributively and asynchronously among sources because the total utility monotonically increases each time θ_s is updated and the information required for updates is just the local information of source s.

Remark 1: The α-fair utility family for single-path has the form of

$$U(x) = \begin{cases} \log(x), & \text{if } \alpha = 1, \\ \frac{x^{1-\alpha}}{1-\alpha}, & \text{if } \alpha \in (0,1) \cup (1, \infty). \end{cases} \tag{14}$$

Assuming $\mathbf{x}^*$ be a global optimum of Main Problem, the inequality $\nabla G(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) \leq 0$ always holds for all feasible point $\mathbf{x}$ because of the concavity of $G(\mathbf{x})$. Therefore,

$$\sum_{s=1}^{N} \sum_{i=1}^{R_s} \frac{x_{s,i} - x_{s,i}^*}{(\sum_{j=1}^{R_s} x_{s,j}^*)^\alpha} = \sum_{s=1}^{N} \frac{x_s - x_s^*}{(x_s^*)^\alpha} \leq 0, \tag{15}$$

where $x_s = \sum_{j=1}^{R_s} x_{s,j}$. So we still have the α-fairness among sources in a multi-path environment. For example, we have proportional fairness with $\alpha = 1$, harmonic mean fairness with $\alpha = 2$, and max-min fairness with $\alpha = \infty$ [14].

III. Multi-path TCPs

In this section, we want to develop a design framework of Multi-path TCPs (MTCPs) base on the above theoretical analysis. There are three main targets that we want to follow when designing MTCPs:

1) MTCPs must be compatible to the current single-path TCPs. The multi-path users can run simultaneously with the single-path users which are using the current TCPs in a same network.
2) The model must address the mismatch parameters among paths from one source, such as different round-trip-time in Reno.
3) The protocols can be implemented online.

From the work of S. Low [12], we know that the current TCPs are the implicit solutions to the NUM problems with particular utility functions. All of these functions are concave functions. Thus, we can apply our approximation inequality (1) to the NUM's objective. The approximation problem is exactly the basic NUM which each path is treated as a single-path flow associated with a new strictly concave utility parameterized by $\theta_{s,i}$. However, it is clear that the second target cannot be satisfied because Algorithm 1 does not address the specific parameters for each route. On the other hand, to address the first target, the function $\tilde{U}(.)$ should have the form similar to the utility functions of the current TCPs. The $\boldsymbol{\theta}$ update in MTCPs should become 1 in case of single-path users. And with $\theta = 1$ for single-path users, the utility function, rate update, and window change of MTCPs should become exactly the ones of the single-path users. Therefore, we cannot apply the approximation inequality (1) directly, we use the modified approximation coefficient $\hat{\theta}$ instead of θ.

A notification in online implementation is the assumption that the approximation problem is solved in T iterations and we can choose T large enough for the convergence of every approximation problems. After T iterations, $\boldsymbol{\theta}$ is updated. The following part is an example of deploying our theoretical framework to design a new multi-path Reno, which are compatible to TCP Reno. However, our frameworks can be applied to design any multi-path TCP which the corresponding TCP has an implicit concave utility function.

A. Multi-path Reno (mReno)

The utility function of Reno for single-path user is given by

$$U_s(x_s) = \frac{\sqrt{3/2}}{D_s} \tan^{-1}\Big(\sqrt{\frac{2}{3}} D_s x_s\Big), \tag{16}$$

where D_s is RTT of the path.[1] We construct the utility function for multi-path users as follows

$$U_s(\mathbf{x}_s) = \frac{\sqrt{3/2}}{D_s^{\min}} \tan^{-1}\Big(\sqrt{\frac{2}{3}} D_s^{\min} \sum_{i=1}^{R_s} x_{s,i}\Big), \quad (17)$$

where $D_s^{\min}$ is the minimum RTT over all paths from source s. In order for $\tilde{U}(.)$ to have the similar form to (16), $\theta_{s,i}$ is chosen such that

$$\theta_{s,i} = \frac{x_{s,i}}{\sum_{i=1}^{R_s} x_{s,i}} = \frac{D_s^{\min}}{D_{s,i}} \hat{\theta}_{s,i}$$

or

$$\hat{\theta}_{s,i} = \frac{D_{s,i}}{D_s^{\min}} \frac{x_{s,i}}{\sum_{i=1}^{R_s} x_{s,i}} \quad (18)$$

The approximation inequality becomes

$$\begin{aligned} U_s(\mathbf{x}_s) &\geq \frac{\sqrt{3/2}}{D_s^{\min}} \sum_{i=1}^{R_s} \theta_{s,i} \tan^{-1}\Big(\sqrt{\frac{2}{3}} D_s^{\min} \frac{x_{s,i}}{\theta_{s,i}}\Big) \\ &= \sum_{i=1}^{R_s} \frac{\sqrt{3/2}}{D_{s,i}} \hat{\theta}_{s,i} \tan^{-1}\Big(\sqrt{\frac{2}{3}} \frac{D_{s,i} x_{s,i}}{\hat{\theta}_{s,i}}\Big) \\ &\triangleq \sum_{i=1}^{R_s} \tilde{U}_{s,i}(x_{s,i}; \hat{\theta}_{s,i}). \end{aligned} \quad (19)$$

We get the following equation

$$q_{s,i} = \frac{3}{2x_{s,i}^2 \frac{D_{s,i}^2}{\hat{\theta}_{s,i}^2} + 3} \quad (20)$$

at the equilibrium point and from (20) and we construct the rate update of mReno in one time slot as follows

$$x_{s,i}(t+1) = \Big[x_{s,i}(t) + \hat{\theta}_{s,i}^2 \frac{1 - q_{s,i}(t)}{D_{s,i}^2} - \frac{2}{3} q_{s,i}(t) x_{s,i}^2(t)\Big]^+. \quad (21)$$

In one round-trip-time, the window size of each subflow in mReno increases by $\hat{\theta}_{s,i}^2$ if an ACK is received on that subflow and decreases by half if it is not.[2]

In the above analysis, we can choose $D_s^{\max}$ or D_s^{aver} instead of $D_s^{\min}$. Whichever value we choose, it plays the compatibility role for the multi-path TCP to the single-path flows, therefore, the single-path TCP does not need to change. On the other hand, using $D_s^{\min}$ means that the network will treat the multi-path user as a single-path user on the path with the minimum round-trip-time. This also keeps fairness among single-path users and multi-path users. This is quite a reasonable thinking in multi-path routing.

IV. mReno Implementation

We use both NS-2 and Matlab to implement the experiments. The approximation problem is solved in T=100 inner iterations. After T iterations, $\boldsymbol{\theta}$ is updated. In packet level simulation, we build our code based on the open source code of IETF's MPTCP on NS-2 [15]. Random Early Detection algorithm is used. Algorithm 1 is run in Matlab to calculate the theoretical optimal solution. In Matlab environment, the round-trip-time parameter in the utility function as well as in the rate update is approximated to double of propagation delay. The stopping criterion is $\left|\frac{\mathbf{x}(t)-\mathbf{x}(t-1)}{\mathbf{x}(t-1)}\right| \prec \epsilon$ and $\left|\frac{\mathbf{x}(\tau)-\mathbf{x}(\tau-1)}{\mathbf{x}(\tau-1)}\right| \prec \epsilon$ for both inner and outer iterations, where $\epsilon = 10^{-5}$. The diminishing stepsize $\kappa = 5 \times 10^{-4}/t$ is used.

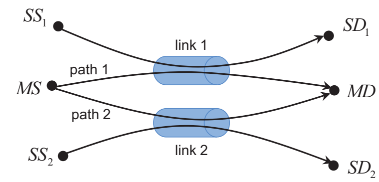

Fig. 1. Network with two bottleneck links.

The network has two links and three users, one multi-path user and two single-path users as shown in Figure 1. The propagation delay of all paths are 50ms in the first case and 50ms for link 1 and 200ms for link 2 in the second case. The capacities of links are 4Mbps. We observe the network for 1000 seconds. The users are on/off as follows:

1) 0s-200s: only multi-path user running,
2) 200s-400s: multi-path user and one single-path user running,
3) 400s-800s: multi-path user and both single-path users running, and
4) 800s-1000s: only multi-path user running.

Figures 2(a-d) show the rate allocation of users in cases of same round-trip-time and different round-trip-time. All the plots seem to follow the theoretical results (the thick-dot lines). In case of same round-trip-time, the rate of multi-path user is always similar to the rates of single part users in both phases 2 and 3, Figure 2(a). From Figure 2(b) we can see that when the single-path user 1 is on, the traffic of multi-path user is shifted to link 2 so link 1 is for the single-path user 1 traffic to get the fairness. In case of different round-trip-time, we can see in phase 3 of Figures 2(c) that the rate of multi-path user is similar to the rate of the single-path user 1, which is on the shorter RTT path. This result agrees with the analysis, network treats multi-path user as a single-path user on the path with minimum RTT.

V. Conclusions

We have solved the multi-path NUM by using the successive approximation method. The proposed algorithm converges to a global optimal solution of the problem. Also utilizing the reverse engineering framework of TCP Reno, we design the corresponding multi-path Reno which can coexist and be compatible to Reno. A single-path user running mReno reacts like running TCP Reno. Our protocol also addresses the case of mismatch RTT among subflows. The simulations on mReno

[1] If the round-trip-time is very large, then the utility function of Reno $U_s(x_s) \approx -\frac{3}{2x_s D_s^2}$, a utility in the α-fairness family with $\alpha = 2$ [12].

[2] The factor 1/2 is precisely replaced by 2/3 when describing the TCP behavior in the mathematical model.

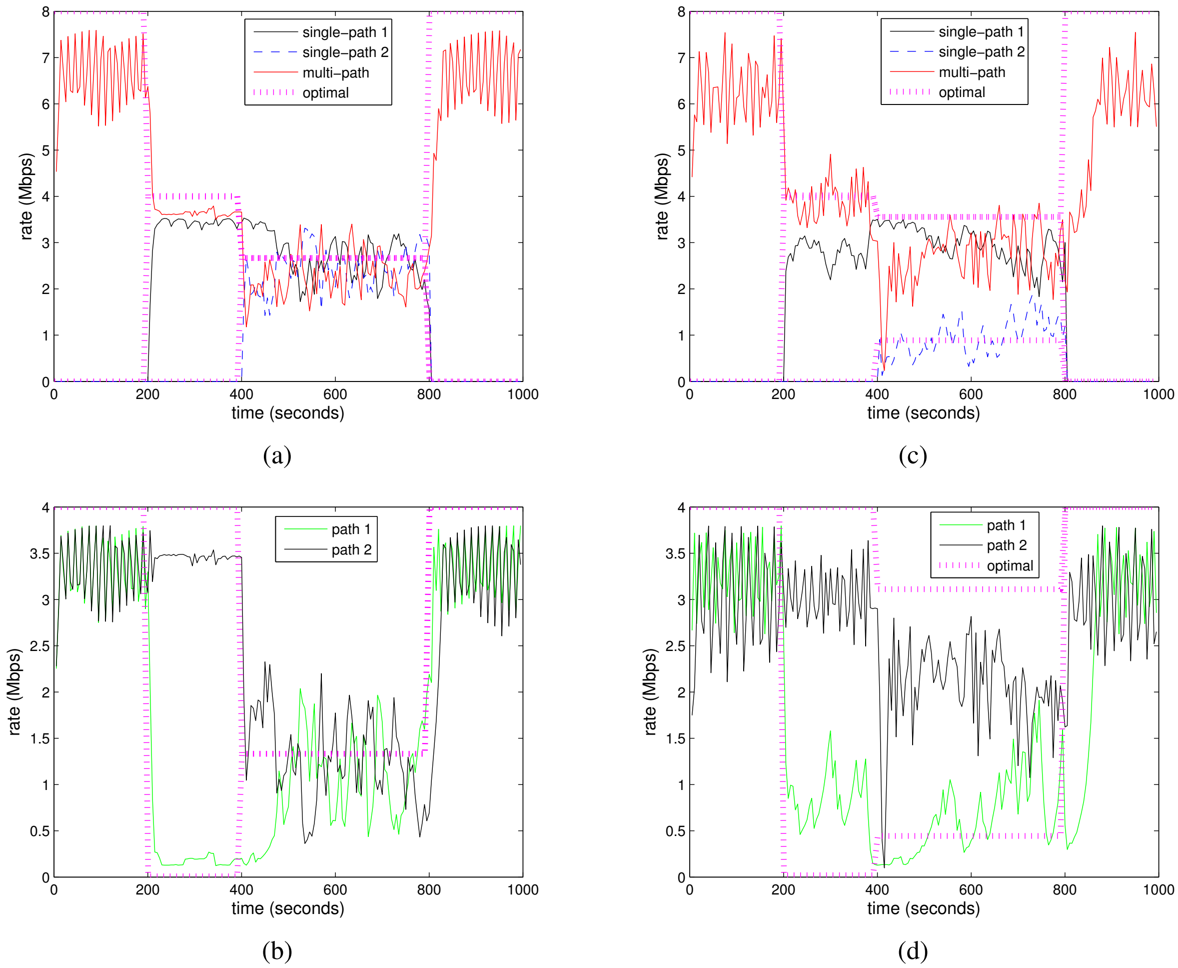

Fig. 2. Rate of flows and subflows: (a-b) same RTTs; (c-d) different RTTs.

show the fairness among the multi-path and single-path users in the network.

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